GENERALIZED IMPEDANCE CIRCLE DIAGRAMS IN THE ANALYSIS OF COUPLED NETWORKS

By CHANDRASEKHAR GHOSH

(Received for publication, March 11, 1950)

ABSTRACT. The use of circle diagrams in the analysis of circuit behaviour in electrical engineering problems is well-known When applied to the determination of circuit behaviour with reference to variation in the impedance of circuits, the impedance circle diagram is very useful in the solution of network problems where mutual impedances are involved The analysis of circuit behaviour by considering the complete circuit as an equivalent T-section leads to a very useful way of visualizing the performance with the help of circle diagrams. In this paper the form of relation leading to circular loci for the impedance is developed and a practical method of developing the impedance circle diagrams for a representative T-section under different conditions of variation in the impedance is shown

In electrical engineering, steady-state impedances, admittances, voltages and currents in simple series and parallel circuits are studied from the analytical point of view Graphical representations, in the form of vector diagrams are of considerable help in the visualization of the analytical expressions.

In the simple plots of steady-state time vectors, the locus of the terminal of any individual rotating vector is a circle concentric with the origin and the angular displacement of the vector at any instant is proportional to the independent variable, time. The use of this vector diagram can be extended to include a range of steady-state conditions by letting a vector sweep out a locus in the complex plane as either a parameter or the frequency is varied. This application differs from the first concept in that the locus is not swept out by a *time* vector. By means of this extended vector diagram, the circuit behaviour can be visualized and quantitatively analyzed not only for a single steady-state condition but also for a range of conditions. This method is especially convenient and practically useful because these loci prove to be circles (or straight lines) in many useful cases. For these reasons such loci or *circle diagrams*, as they are commonly called, are widely used in both the power and the communication fields.

In problems on transmission of power, the entire transmission system, used to connect an electrical generating station with a distant load centre including a long tansmission line and transformers at both ends, is usually analyzed as an *l*-loop network with the generating station as one source and the load as an equivalent of a negative source. Such an *l*-loop network with two pairs of terminals, one pair in each of two loops, to which sources (or loads which, from the point of view of the l-loop network, are treated as sources) can be connected is often called a coupling network.

The coupling network is often rather complex and, in a broad use of the term, may, as mentioned above, contain long transmission lines or even energy-conversion devices, so that between the two pairs of terminals a mechanical, accoustical, or electric-wave link may exist In the latter cases the equivalent electrical behaviour of the mechanical or other link must be known before the combination of such a link and electric circuits can be treated as a coupling or two-terminal pair network.

Thus considering a two-terminal-pair network, as shown in Fig. 1, the current and voltage relations at the terminals 1-1′ and 2′-2 are given by

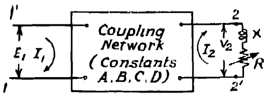

FIG. 1
Coupled net work

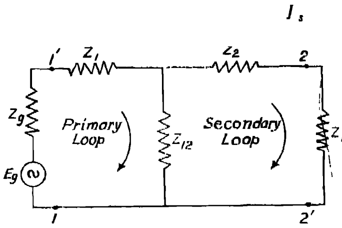

FIG 2
T-section net work

$$E_1 = Z_{11} I_1 + Z_{12} I_2 \qquad \text{.. (1)}$$
$$V_2 = Z_{12} I_1 + Z_{22} I_2$$

where Z_{11} = impedance in loop 1
Z_{22} = impedance in loop 2
Z_{12} = mutual impedance between loops 1 and 2

For power engineering it is usually convenient to express the voltage and current at one end of a two-terminal-pair network in terms of the voltage and current at the other end. Such equations are much used by the power engineer in preference to the equations giving the two voltages in terms of the two currents and *vice versa*.

The equations of the coupling network as used in power circuits are customarily written

$$E_1 = A V_2 + B I_2 \qquad \text{.. (2)}$$
$$I_1 = C V_2 + D I_2$$

in which the parameters A, B, C and D are called *general circuit constants*, of which only three are independent. These constants are related to the Z's in the former equations as follows

$$A = -\frac{Z_{11}}{Z_{12}};\ B = -\frac{D_Z}{Z_{12}};\ C = -\frac{1}{Z_{12}};\ D = -\frac{Z_{22}}{Z_{12}}$$

where Dz is the determinant of Z's, *i.e.*

$$Dz = \begin{vmatrix} Z_{11} & Z_{12} \\ Z_{12} & Z_{22} \end{vmatrix} = Z_{11}\,Z_{22} - Z^2_{12}$$

Considering now the branch containing the variable resistance R and constant reactance X as the load Z_L, if it is desired to determine the circle diagram of the current I_1, as the resistance of any other branch is varied over a definite range, then the input impedance

$$Z_{1a} = \frac{E_1}{I_1} = \frac{AV_2 + BI_2}{CV_2 + DI_2}$$

$$= \frac{A\left(\frac{V_2}{I_2}\right) + B}{C\left(\frac{V_2}{I_2}\right) + D} = \frac{AZ_L + B}{CZ_L + D} \qquad \dots \quad (3)$$

Since $Z_L = R + jX$, in which R is variable

$$Z_{1a} = \frac{A(R + jX) + B}{C(R + jX) + D}$$

$$= \frac{AR + (jAX + B)}{CR + (jCX + D)} \qquad \dots \quad (3a)$$

Similarly, the input admittance

$$Y_{1a} = \frac{CZ_L + D}{AZ_L + B} = \frac{CR + (jCX + D)}{AR + (jAX + B)} \qquad \dots \quad (3b)$$

Both these expressions for Z_{1a} and Y_{1a} are in the general form of

$$F(\rho) = \frac{M(\rho) + N}{T(\rho) + U} \qquad \dots \quad (4)$$

in which M, N, T and U are complex constants. The locus of $F(\rho)$, as shown by Schumann (1922), is a circle. Hence the input impedance or admittance described by these expressions can be represented by a circular locus in the complex plane.

All such networks involving mutual impedance can be simplified into an equivalent T-section. The circle diagrams used for analytical work in such cases are usually referred to as *impedance circle diagrams*. Such diagrams are very helpful in various problems in connection with network solutions and also in relay work. The use of such diagrams show properties of the original circuit that might be overlooked without their use. As already noted, the impedance circle diagram is adaptable not only to problems in transmission and relaying, but also to problems relating to coupled circuits, electromechanical problems and even to any energy-conversion devices.

Thus taking the general case of a T-section network, shown in Fig. 2, the looking-in impedance, *i.e.*, the impedance as on looking into end 1 is

$$Z_{LI}=Z_1+\frac{(Z_2+Z_L)Z_{12}}{Z_2+Z_L+Z_{12}} \tag{5}$$

Put in a more convenient form

$$Z_{LI}=\left\{Z_1+Z_{12}\right\}+\left\{\frac{(Z_2+Z_L)Z_{12}}{Z_2+Z_{12}+Z_L}-Z_{12}\right\}$$

$$=\left\{Z_1+Z_{12}\right\}-\frac{Z^2{}_{12}}{Z_2+Z_{12}+Z_L} \tag{6}$$

But as shown in Fig. 1, $Z_1+Z_{12}=Z_{11}$ the total impedance of the primary loop of the T-section, and $Z_2+Z_{12}+Z_L=Z_{22}$' the total impedance of the secondary loop. Hence we have

$$Z_{LI}=Z_{11}-\frac{Z_{12}{}^2}{Z_{22}}$$

$$=\frac{Z_{11}Z_{22}-Z_{12}{}^2}{Z_{22}} \quad \ldots \quad (6a)$$

This again is one of the particular forms of the general equation for a circle. The locus of Z_{LI} is therefore a circle under all conditions.

The construction and geometry of the impedance circle diagrams have been discussed by Stewart (1944). The construction of the impedance circle diagrams can, however, be modified to make it more convenient for use in problems concerned with coupled circuits and in relaying. In determining the limits of operation of relays one of the important requirements is to anticipate the possible variation in the looking-in impedance, as viewed by the relay, and thus to be able to provide the proper setting for the relay.

In relay operations the possible variations in the circuit constants to be anticipated are in the value of resistance, reactance and/or power-factor angle. The modified relations for the circles with reference to these conditions are now discussed.

Case I

Take Z_{22}=constant. Further let θ_{12}=power factor angle of the mutual impedance, and θ_{22}=power factor angle of the secondary loop impedance

Then $$Z_{LI}=Z_{11}-\frac{|Z^2{}_{12}|}{|Z_{22}|}\underline{/2\theta_{12}-\theta_{22}} \quad \ldots \quad (7)$$

$$=Z_{11}+\frac{|Z^2{}_{12}|}{|Z_{22}|}\underline{/180^\circ+2\theta_{12}-\theta_{22}} \quad \ldots \quad (7a)$$

The locus of Z_{LI} is a circle with the centre at the end point of vector Z_{11} and radius equal to $\dfrac{|Z^2_{12}|}{|Z_{22}|}$, the angle parameter of the vector Z_{LI} starts from zero position at angle $180^\circ + 2\theta_{12}$ and rotates in a negative direction.

Case II

Take R_{22}=constant; θ_{22} varies

Then
$$Z_{LI} = Z_{11} - \frac{|Z^2_{12}|}{|Z_{22}|} \angle 2\theta_{12} - \theta_{22}$$

$$= Z_{11} - \frac{|Z_{12}{}^2|}{R_{22}} \cdot \frac{R_{22}}{|Z_{22}|} \angle 2\theta_{12} - \theta_{22} \qquad \ldots \quad (8)$$

But $\dfrac{R_{22}}{|Z_{22}|} = \cos\theta_{22}$

Hence
$$Z_{LI} = Z_{11} - \frac{|Z_{12}{}^2|}{R_{22}} \cos\theta_{22} \angle 2\theta_{12} - \theta_{22} \qquad \ldots \quad (8a)$$

$$= Z_{11} - \frac{|Z^2_{12}|}{R_{22}} [\cos\theta_{22} \, [\cos(2\theta_{12} - \theta_{22}) + j \sin(2\theta_{12} - \theta_{22})]]$$

$$= Z_{11} - \frac{|Z^2_{12}|}{R_{22}} \cdot \tfrac{1}{2} \, [\{\cos 2\theta_{12} + j \sin 2\theta_{12}\} + \{\cos(2\theta_{12} - 2\theta_{22}) + j \sin(2\theta_{12} - 2\theta_{22})\}]$$

$$= Z_{11} - \left[\frac{|Z^2_{12}|}{2R_{22}} \angle 2\theta_{12} + \frac{|Z^2_{12}|}{2R_{22}} \angle 2\theta_{12} - 2\theta_{22}\right]$$

$$= Z_{11} + \frac{|Z^2_{12}|}{2R_{22}} \angle 180^\circ + 2\theta_{12} + \frac{|Z_{12}{}^2|}{2R_{22}} \angle 180^\circ + 2\theta_{12} - 2\theta_{22} \qquad \ldots \quad (8b)$$

This is again a circle with centre at $Z_{11} + \dfrac{|Z^2_{12}|}{2R_{22}} \angle 180^\circ + 2\theta_{12}$ and radius equal to $\dfrac{|Z^2_{12}|}{2R_{22}}$, the angle starts from $180^\circ + 2\theta_{12}$.

Case III.

Let X_{22}=constant

Then
$$Z_{LI} = Z_{11} - \frac{|Z^2_{12}|}{|Z_{22}|} \angle 2\theta_{12} - \theta_{22}$$

$$= Z_{11} - \frac{|Z^2_{12}|}{|X_{22}|} \cdot \frac{|X_{22}|}{|Z_{12}|} \angle 2\theta_{12} - \theta_{22} \qquad \ldots \quad (9)$$

$$= Z_{11} - \frac{|Z^2_{12}|}{|X_{22}|} \cdot \sin\theta_{22} \angle 2\theta_{12} - \theta_{22} \qquad \ldots \quad (9a)$$

$$= Z_{11} - \frac{|Z_{12}^2|}{|X_{22}|}\left[\sin\theta_{22}\cos(2\theta_{12}-\theta_{22}) + j\sin\theta_{22}\sin(2\theta_{12}-\theta_{22})\right]$$

$$= Z_{11} - \frac{|Z_{11}^2|}{2|X_{22}|}\,\underline{/2\theta_{12}-90^\circ} - \frac{|Z_{12}^2|}{2|X_{22}|}\,\underline{/2\theta_{12}-2\theta_{22}+90^\circ}$$

$$= Z_{11} + \frac{|Z_{12}^2|}{2|X_{22}|}\,\underline{/2\theta_{12}+90^\circ} + \frac{|Z_{12}^2|}{2|X_{22}|}\,\underline{/2\theta_{12}-2\theta_{22}+90^\circ} \qquad \ldots \quad (9b)$$

The locus is a circle the centre of which is at $Z_{11} + \frac{|Z_{12}^2|}{2|X_{22}|}\,\underline{/2\theta_{12}+90^\circ}$ and radius $\frac{|Z_{22}^2|}{2|X_{22}|}$, the angle starts with zero at $2\theta_{12}+90^\circ$

Case IV

Let θ_{22} = constant

Then $$Z_{LI} = Z_{11} - \frac{|Z_{12}^2|}{|Z_{22}|}\,\underline{/2\theta_{12}-\theta_{22}}$$

$$= Z_{11} + \frac{|Z_{12}^2|}{|Z_{22}|}\,\underline{/2\theta_{12}-\theta_{22}+180^\circ} \qquad \ldots \quad (10$$

Thus the locus is a circle with centre at infinity and radius infinite, *i e.*, a straight line inclined to the R-axis at an angle $2\theta_{12}+180^\circ$, and passing through Z_{11} when $Z_{22}=\theta$, *i e.* open circuit.

Construction of Impedance Circle Diagrams for the above cases.

In all the above cases, the second term in the expression for Z_{LI} is of the form a^2/b, where a stands for Z_{12} in all the cases while b stands for Z_{22} in cases I and IV, for $2R_{22}$ in case II, and for $2X_{22}$ in case III. Hence, the

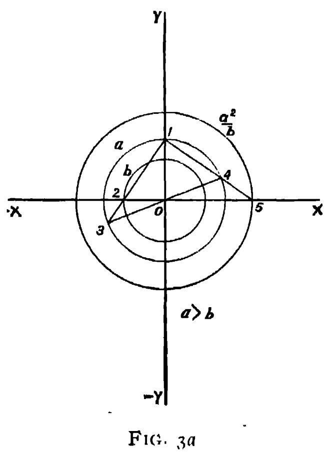

FIG. 3a

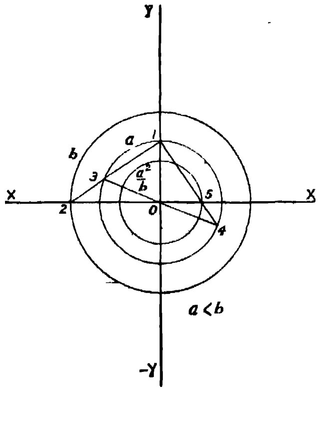

FIG. 3b

Determination of a^2/b

starting point in obtaining the impedance circle diagram is determining the magnitude of the expression for the radius. A very convenient graphical method of determining the magnitude of such an expression is indicated in Figs. 3*a* and 3*b*. Fig. 3*a* is for the case when $a > b$ and Fig. 3*b* is for the case when $a < b$. The successive steps are: join 1 on '*a*' circle with 2 on '*b*' circle to get the point 3 on the '*a*' circle; join 3 and the origin of co-ordinates and produce it to obtain point 4 on the '*a*' circle. Join 1 and 4 and obtain the point 5 on the X-axis. Then 0-5 is the measure of a^2/b, the radius required.

On the normal rectangular co-ordinate paper first plot Z_{11} and draw a circle with the origin as centre and radius equal to a^2/b. Then in terms of the arc on the a^2/b circle, the angle $2\theta_{12}+180°$ in cases I and II, or the angle $2\theta_{12}+90°$ in case III is measured, the line through this angle gives the direction of the datum line The datum line is next drawn parallel to this line through the end point of vector Z_{11}.

Case I.

The circle of radius a^2/b, *i.e.* $\dfrac{|Z_{12}^2|}{|Z_{22}|}$ drawn with the end point of vector

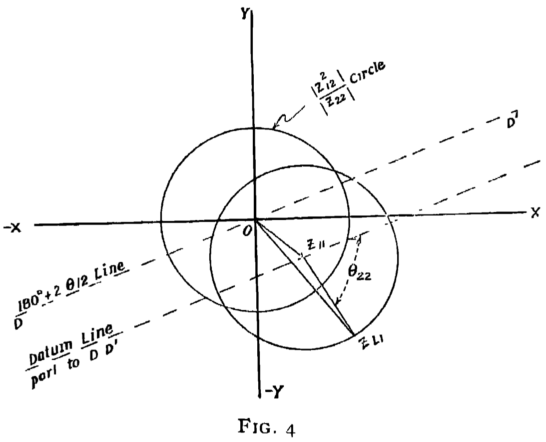

FIG. 4

Case I Z_{22} = constant

Z_{11} as centre, is the looking-in impedance circle for case I. θ_{22} is measured clockwise from the datum line, to obtain the value of Z_{LI}. This is shown in Fig. 4.

Cases II and III

The point of intersection of the datum line through the end point of vector Z_{11} and the second a^2/b circle drawn with Z_{11} as centre, becomes

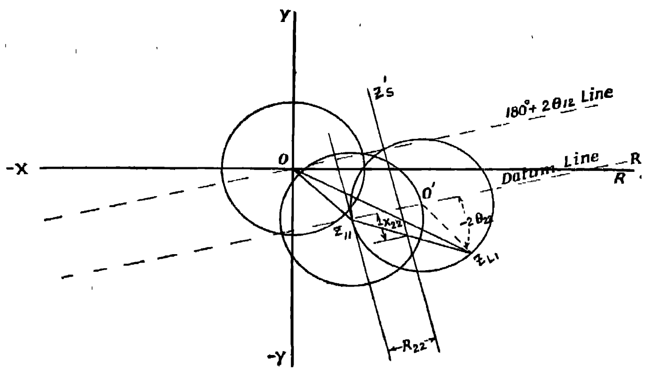

FIG. 5
Case II. R_{22}=constant

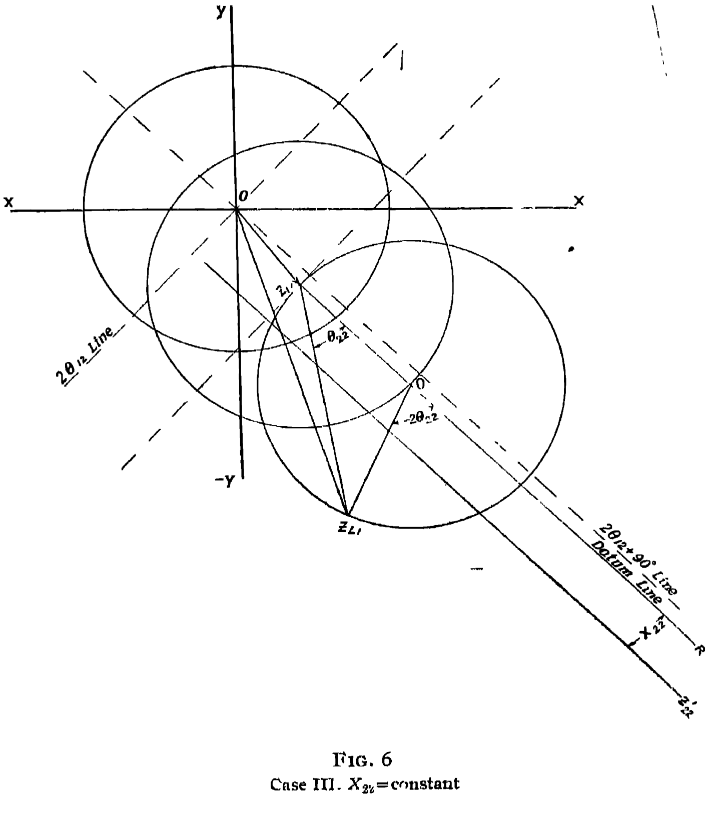

FIG. 6
Case III. X_{22}=constant

the centre of the required impedance circle Z_{LI}. With the same datum line as the zero angle line, the angle $2\theta_{22}$ is measured in the negative (*i.e.* clockwise) direction on the third circle. This point on the third circle, thus obtained, locates the end of the vector for the looking-in impedance Z_{LI}. These are shown in Figs. 5 and 6.

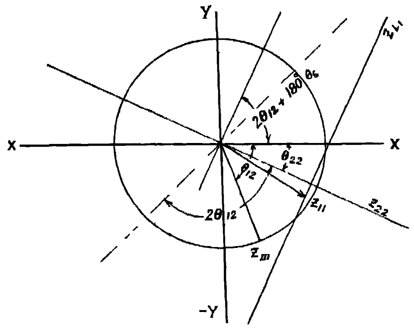

FIG. 7

Case IV. θ_{22} = constant

The end point of the vector Z_{LI} can also be located by plotting the locus of Z_{22} in the new co-ordinates, joining Z_{11} with the proper point on the Z_{22} locus and producing this line to meet the third circle. This meeting point with the third circle locates the end point of the vector Z_{LI}. This is also shown in the figures.

Case IV

As already discussed above, the locus of Z_{LI}, in the event θ_{22} is a constant, is a straight line, that is, a circle with infinite radius. This is shown in Fig. 7.

Further analytic work on the above line is being carried on with different types of actual circuit conditions. These will be communicated later.

DEPARTMENT OF POWER ENGINEERING
INDIAN INSTITUTE OF SCIENCE
BANGALORE

REFERENCES

Schumann, W. O., 1922, *Arch. f. Elek.*, **11**, 140-46.
Stewart, H. L., 1944, *General Electric Review*, **47**, 20-25.